

A Reducing of the Invariant Semidefinite Subspace Problem for Kreĭn Noncontraction to such a Problem for Kreĭn Isometry

S.A. Chorošavin

Abstract

Theorem. If every J -isometry has nontrivial positive invariant subspace then every J -noncontraction has such a subspace.

Theorem. If every J -binoncontractive J -isometry has maximal positive invariant subspace then every J -noncontraction has such a subspace.

0.

Let H be a Hilbert space equipped with the fixed orthogonal decomposition $H = H_- + H_+$. The indefinite inner product

$$x, y \in H \mapsto \langle x, y \rangle \in C$$

is introduced now by the formula

$$\langle x, y \rangle := (x, Jy); \quad J := P_- + P_+;$$

where $(,)$ is the symbol of the usual Hilbert scalar product and P_{\pm} are the orthoprojectors of H onto H_{\pm} . The pair H, \langle , \rangle is said to be *Krein space* or *J -space*.

A linear bounded $T : H \rightarrow H$ is said to be *J -noncontraction* or \langle , \rangle -noncontraction or *Krein noncontraction*, iff

$$\langle Tx, Tx \rangle \geq \langle x, x \rangle \quad (x \in H)$$

Particular cases of Krein noncontraction are *J-isometry* ($\langle \cdot, \cdot \rangle$ -isometry, *Krein-isometry*):

$$\langle Vx, Vx \rangle = \langle x, x \rangle \quad (x \in H)$$

Krein-unitary:

$$\langle Ux, Ux \rangle = \langle x, x \rangle, \quad \langle U^\dagger x, U^\dagger x \rangle = \langle x, x \rangle \quad (x \in H)$$

and *Krein binoncontraction*:

$$\langle Tx, Tx \rangle \geq \langle x, x \rangle, \quad \langle T^\dagger x, T^\dagger x \rangle \geq \langle x, x \rangle \quad (x \in H)$$

hereinafter T^\dagger is $\langle \cdot, \cdot \rangle$ -adjoint of T (Note, $T^\dagger = JT^*J$, where T^* is the Hilbert adjoint of T).

The traditional question of the theory of linear operators is the question of the existense of a (nontrivial) invariant subspace. In the case of indefinite inner product spaces it is of interest a special kind of subspaces:

A subspace $L \subset H$ is said to be *positive* (*negative*) iff

$$\langle x, x \rangle \geq 0 \quad (\leq 0) \quad \text{for any } x \in L.$$

A subspace $L \subset H$ is said to be *maximal positive* (*negative*) iff it is positive (*negative*) and maximal through such of subspaces in the sence of the set theory.

In order that the subspace $L \subset H$ should be maximal positive, a necessary and sufficient condition is that there should exist a linear $K : H_+ \rightarrow H_-$ such that

$$\|K\| \leq 1 \text{ and } L = \{x_+ + Kx_+ | x_+ \in H_+\}.$$

Such an operator K is said to be *angular operator* of L and it is unique (by fixed L). The structure of cloused positive subspace L is analogous: $L = \{x_+ + Kx_+ | x_+ \in L_+\}$, where L_+ is a cloused linear subspace of H_+ and $K : L_+ \rightarrow H_-$ is linear with $\|K\| \leq 1$.

We are interested in the existence of maximal positive invariant subspace for Krein noncontraction.

At first sight the question for noncontraction seems to be more general than the one for isometry and the last question seems to be more general than the question for Krein unitary operator. We shall show that it is not entirely the case.

1.**Theorem 1.**

If every Krein isometry has a nontrivial positive invariant subspace (resp. maximal positive invariant subspace), then every Krein noncontraction has such a subspace.

PROOF. Let us consider a denumerable Hilbert direct sume

$$\hat{H} := H \oplus H \oplus \dots = \bigoplus_{n=1}^{\infty} H$$

with the canonical projection $p := \hat{H} \rightarrow H$

$$p : x_1 \oplus x_2 \oplus \dots \mapsto x_1$$

canonical embedding $j : H \rightarrow \hat{H}$

$$j : x \mapsto x \oplus 0 \oplus 0 \dots$$

and introduce operators $\hat{P}_{\pm} : \hat{H} \rightarrow \hat{H}$, $\hat{J} : \hat{H} \rightarrow \hat{H}$ by the formulas:

$$\hat{P}_+ := P_+ \oplus 0 \oplus 0 \oplus \dots$$

$$\hat{P}_- := P_- \oplus I \oplus I \oplus \dots$$

$$\hat{J} := J \oplus -I \oplus -I \oplus \dots$$

Note: the operators \hat{P}_{\pm} are orthoprojections, $\hat{P}_+ + \hat{P}_- = I_{\hat{H}}$, $\hat{J} = \hat{P}_+ + \hat{P}_-$, and \hat{H} is a Krein space with respect to the decomposition

$$\hat{H} = \hat{H}_+ + \hat{H}_-, \quad \hat{H}_{\pm} := \hat{P}_{\pm} \hat{H}$$

Let \ll, \gg will denote the correspondent indefinite inner product on \hat{H} . We have to remark the next properties of it:

1) if $\ll x, x \gg \geq 0$ then $\ll px, px \gg \geq 0$; ($x \in \hat{H}$)

2) if $\ll x, x \gg \geq 0$ and $x \neq 0$ then $px \neq 0$; ($x \in \hat{H}$)

and hence if L is a nontrivial subspace of \hat{H} , then the pL is a nontrivial positive subspace of H . Moreover if L is a maximal positive subspace of \hat{H}

then the pL is a maximal positive subspace of H . To show this it is sufficient to observe that $p\hat{P}_\pm = P_\pm, jP_+ = \hat{P}_+$. Hence if L is a maximal positive with the angular operator $K : \hat{H}_+ \rightarrow \hat{H}_-$:

$$L = \{x_+ + Kx_+ | x_+ \in \hat{H}_+\}$$

then

$$\begin{aligned} pL &= \{px_+ + pKx_+ | x_+ \in \hat{H}\} \\ &= \{\tilde{x}_+ + pKj\tilde{x}_+ | \tilde{x}_+ \in H_+\}, \end{aligned}$$

$\|pKj\| \leq \|K\| \leq 1$ and pL is maximal positive with the angular operator pKj .

Now the rest is fast evident:

If T is a Krein noncontraction, then $T^*JT - J \geq 0$ and there exists the Hilbert square root $D = (T^*JT - J)^{1/2}$. Constructing the operator $V : \hat{H} \rightarrow \hat{H}$ as follows

$$V : x_1 \oplus x_2 \oplus x_3 \oplus \dots \mapsto Tx_1 \oplus Dx_1 \oplus x_2 \oplus x_3 \oplus \dots$$

we obtain

$$Tp = pV, \quad \ll Vx, Vx \gg = \ll x, x \gg \quad (x \in \hat{H})$$

Hence V is a Krein isometry and if a subspace L is invariant for V then the pL is invariant for T . \square

\square

Remark. If $\dim H_\pm = \infty$ both, then \hat{H} is isomorph to H as a Krein space. Hence for this case the theorem 1 can be reformulated so : If every Krein isometry $V : H \rightarrow H$ has a nontrivial positive invariant subspace (resp. maximal positive invariant subspace), then every Krein noncontraction $T : H \rightarrow H$ has such a subspace. \square

2.

Theorem 2.

If every Krein binoncontractive Krein isometry has a maximal positive invariant subspace, then every Krein isometry has such a subspace.

PROOF. Let $\hat{H} := H \oplus H$ be the usual Hilbert direct sum with canonical projection $p : \hat{H} \rightarrow H$ defined by the formula

$$p : x_1 \oplus x_2 \mapsto x_1 \quad (x_1, x_2 \in H)$$

and let $\hat{J}, \hat{V} : \hat{H} \rightarrow \hat{H}$ be linear operators defined as follows: $\hat{J} := J \oplus I$,

$$\hat{V} := \begin{pmatrix} V & A \\ 0 & B^* \end{pmatrix}$$

where $V : H \rightarrow H$ is a Krein isometry and $A, B : H \rightarrow H$ are linear and bounded. The space \hat{H} is the Krein space with respect to \hat{J} and if L is a maximal positive subspace of \hat{H} then $L' := p(L \cap H \oplus \{0\})$ is a maximal positive subspace of the space H . If L is invariant for \hat{V} , then L' is invariant for V .

We shall demonstrate that there exist A and B such that the operator \hat{V} is a Krein binoncontractive Krein isometry.

Previously we have to make some observations.

1) $J - VJV^*$ is selajoint and

$$\text{Ran}(J - VJV^*) = \text{Ran}(I - VV^+) = \text{Ker } V^+ = \text{Ker } V^*J;$$

let $J - VJV^* = \int_{-\infty}^{+\infty} \lambda dE_\lambda$ be the spectral decomposition and put

$$p_+ := \int_{+0}^{\infty} dE_\lambda,$$

$$A := (J - VJV^*)_+^{1/2} = \int_{+0}^{\infty} \lambda^{1/2} dE_\lambda$$

So A and p_+ are selfadjoint, positive and p_+ is an orthoprojector onto $\overline{\text{Ran } A}$. Hence

$$\begin{aligned} \text{Ran } p_+ &= \overline{\text{Ran}(J - VJV^*)p_+} \subset \text{Ran}(J - VJV^*) = \text{Ran}(I - VV^\dagger) \\ &= \text{Ker } V^\dagger = \text{Ker } V^*J \end{aligned}$$

2) We have $(JV)^* = V^*J$ and hence

$$H = \text{Ker } V^*J + \text{Ran } JV, \quad \text{Ker } V^*J \perp \text{Ran } JV$$

Let $JV = u|JV|$ be the correspondent polar decomposition where u is the partial isometry with final space $\text{Ran } JV$ and initial space $\text{Ran } V^*J = \text{Ran } V^+ = H$; hence, u is an isometry. But

$$\text{Ran } (I - p_+) = \text{Ker } p_+ \supset (\text{Ker } V^*J)^\perp = \text{Ran } JV$$

hence there exist an isometry of the space H with final space $\text{Ran } (I - p_+)$; we define the operator B as such an isometry.

Now we obtain by immediate computations:

$$\begin{aligned} \hat{V}^* \hat{J} \hat{V} &= \begin{pmatrix} V^*JV & V^*JA \\ AJV & AJA + BB^* \end{pmatrix} = \begin{pmatrix} J & V^*JA \\ AJV & AJA + (I - p_+) \end{pmatrix} \\ \hat{V} \hat{J} \hat{V}^* &= \begin{pmatrix} JV^* + A^2 & AB \\ B^*A^* & B^*B \end{pmatrix} = \begin{pmatrix} JV^* + A^2 & AB \\ B^*A^* & I \end{pmatrix} \end{aligned}$$

Using the identities $A = Ap_+ = p_+A = p_+Ap_+$, $B = (I - p_+)B$ we obtain $V^*JA = V^*Jp_+A = 0$ (and hence $AJV = 0$), $AB = Ap_+(I - p_+)B$ (and hence $B^*A^* = 0$),

$$\begin{aligned} A^2JA^2 &= |J - JV^*|p_+Jp_+|J - JV^*| \\ &= (J - JV^*)p_+Jp_+(J - JV^*) \\ &= p_+(J - JV^*)J(J - JV^*)p_+ \\ &= p_+(J - JV^*)p_+ = p_+|J - JV^*|p_+ \\ &= (|J - JV^*|^{1/2}p_+)p_+ (|J - JV^*|^{1/2}p_+) \\ &= Ap_+A \end{aligned}$$

But A invertible on $\text{Ran } A$, hence $AJA = p_+$ and $\hat{V}^* \hat{J} \hat{V} = p_+$.

Finally we have consequently

$$A^2 = |J - JV^*|p_+ \geq J - JV^*, \quad JV^* + A^2 \geq J, \quad \hat{V} \hat{J} \hat{V}^* \geq \hat{J}$$

and hence \hat{V} is a \hat{J} -binoncontractive \hat{J} -isometry.

□

The article text is the complete text of the author's report on 15-th Voronezh Winter Mathematical School [4]. But in that time the presented constructions and theorems seemed to be rather curious observations. Now the situation is changing (see e.g. LANL E-print math.DS/9908169 or [5], [6])

References

- [1] M. G. Krein : A New Application of the Fixed-Point Principle in the Theory of Operators in a Space with Indefinite Metric. //DAN SSSR. 1964. **154**, N 5, 1023 –1026.(russisch)
- [2] Ju. L. Daletskij, M. G. Krein *The Stability of the Solutions of Differential Equations in a Banach Space*, Moscow.: Nauka, 1970 [Russian]
- [3] M. G. Krein *Introduction to the geometry of indefinite J-spaces and to the theory of operators in those spaces*. In: Second mathematical summer school, Part 1, pp 15-92, Kiev.: Naukova dumka, 1965 [Russian]
- [4] S.A.Chorošavin. J-unitary dilations of bounded operators.(in Russian) // In: 15th Voronezh Mathematical Winter Schcool, 1981, p 119, VINITI 16.12.81, N 5691-81.
- [5] Chorošavin S. A. On one M. G. Krein problem.
//TRANSACTIONS of RANS, series MMMIC, 1997, v.1, N.2, 95-101.
- [6] Chorošavin S. A. An Example of J -Unitary U which Has no Nonzero Invariant Subspace L such that $r(U|L) \leq 1$. //TRANSACTIONS of RANS, series MMMIC. 1998. v.2, N 2, 97–103
- [7] M. Reed , B. Simon, *Functional analysis*, Methods of Modern Mathematical Physics, vol 1, - N.Y.: Academic Press, 1972.